NATURAL MOTION DENSITY OF ELASTIC SHELLS UNDER INTENSIVE DYNAMIC LOADING

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The question of the natural motion density in dynamics problems of elastic shells is considered. Motions are studied for which an exponential growth in amplitude with time occurs. The number of natural motions incident in a given range of variation of the exponent is computed by using an idea of R. Courant. The governing natural motions and condensation points at which the natural motion density tends to infinity are found. The condensation points and governing motions are compared in specific examples.

A natural mode different from the first [1] possesses the greatest rate of growth of deflections for a shell subjected to an intensive dynamic load exceeding the Euler load. The shell deflections tend to infinity with the lapse of time [1, 2]. Finite deformations are developed within a finite time interval in real structures. When the initial system with an infinite number of degrees of freedom in a finite segment approaches a system with a finite number of degrees of freedom, it is necessary to take account of the natural motion density [3].

Let us consider a rectangular shell $0 \le x \le a$, $0 \le y \le b$, hinge-supported along the sides, under zero initial conditions. In the dimensionless variables

$$x_1 = x/a,\ y_1 = y/a,\ 0 \leqslant x_1 \leqslant 1,\ 0 \leqslant y_1 \leqslant b/a,\ a = R_1$$

the motion of a thin-walled elastic shell with constant radii of curvature is described by an equation [4] of the following type, where the subscripts on the x and y have been omitted:

$$\varepsilon^2 \Delta\Delta\Delta\Delta\Phi + \Delta_k\Delta_k\Phi + \lambda\Delta\Delta(\nu\Phi_{xx} + \Phi_{yy}) + R_1^2\rho E^{-1}\Delta\Delta\Phi_{tt} = f(x, y), \tag{1}$$
$$\varepsilon^2 = h^2/12(1-\nu^2)R_1^2, \quad \lambda = hN_2/R_1N_*,$$
$$\Delta_k = (\chi\partial^2/\partial x^2 + \partial^2/\partial y^2), \quad \chi = R_1/R_2,$$
$$\nu = N_1/N_2, \quad N_* = Eh^2R_1^{-1}.$$

Here ε^2 is a small parameter, λ is the overload coefficient, x, y are Cartesian coordinates, $\Phi(x, y, t)$ is the resolving function, $f(x, y)$ is a function determined by small perturbations, h is the thickness, R_1, R_2 are the shell radii of curvature, N_1, N_2 are normal forces in the middle surface along the x, y axes, respectively, N_* is the critical Euler load, ν, E are the Poisson ratio and Young's modulus, ρ is the material density per unit shell surface, and t is the time.

The exact solution of the problem for a hinge-supported shell can be represented as

$$\Phi = \sum_{m,n} q_{mn}(t)\sin k_m x \sin k_n y, \quad k_m = m\pi, \ k_n = n\pi a/b. \tag{2}$$

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The equation obtained for a hinge-supported shell is an asymptotic solution of the problem under arbitrary boundary conditions [4].

Substituting (2) into (1), we obtain an equation for $q_{mn}(t)$ after appropriate manipulations:

$$q''_{mn}(t) - \alpha^2_{mn} q_{mn}(t) = f_{mn}; \qquad (3)$$

$$\alpha^2_{mn} = -\varepsilon^2 (k^2_m + k^2_n)^2 - (\chi k^2_m + k^2_n)^2 (k^2_m + k^2_n)^{-2} + \lambda (\nu k^2_m + k^2_n). \qquad (4)$$

The solution of (3) depends on the values which the coefficient α^2_{mn} takes on. For $\alpha^2_{mn} < 0$ Eq. (3) describes a vibrational process; for $\alpha^2_{mn} = 0$ the process is linear growth of the amplitude q_{mn} with time. Only the case $\alpha^2_{mn} > 0$, will later be considered, i.e., motions for which an exponential increase in the amplitude $q_{mn}(t)$ occurs.

Introducing the new variables r, θ in the k_m, k_n plane,

$$k^2_m + k^2_n = r^2, \quad k^2_m = r^2 \cos^2\theta, \quad k^2_n = r^2 \sin^2\theta,$$

from (4) we obtain

$$\varepsilon^2 r^4 - \lambda r^2 (\nu \cos^2\theta + \sin^2\theta) + (\chi \cos^2\theta + \sin^2\theta)^2 + \alpha^2_{mn} = 0. \qquad (5)$$

Let us use the method of Courant [5] in which the number of eigennumbers $N(\mu)$ less than a given number $\mu*$ is determined approximately as the area of a domain on the m, n plane within which the eigennumber μ is less than the given $\mu*$. The use of this method for different kinds of problems can be found in the survey paper [6]. If the k_m, k_n plane is taken, then the number $N(\alpha^2_{mn})$ is defined as the ratio between the area of the domain in the k_m, k_n within which the exponent α_{mn} is less than a given α_{mn*} and the area of one cell $\Delta k_m \Delta k_n$. We therefore obtain

$$N(\alpha^2_{mn}) = R_1 R_2 2^{-1} \pi^{-2} \int_{\theta_1}^{\theta_2} r^2(\theta)\, d\theta, \qquad (6)$$

where θ_1, θ_2 are the slopes of the radius vector tangent to the domain. In the asymptotic case ($\varepsilon \to 0$), the domain under consideration will be bounded by the coordinate axes k_m, k_n and the curve $r(\theta)$, while the angles θ_1, θ_2 take on the values $\theta_1 = 0$, $\theta_2 = \pi/2$.

The presence or absence of contraction points for the function $N(\alpha^2_{mn})$ is established by investigating the derivative $dN(\alpha^2_{mn})/d\alpha^2_{mn}$. Substituting r^2 from (5) into (6) and differentiating with respect to α^2_{mn}, for the natural motion density we obtain

$$\frac{dN(\alpha^2_{mn})}{d\alpha^2_{mn}} = \frac{R_1 R_2}{4\pi^2 \varepsilon [(1-\chi)^2 - \lambda^2 (1-\nu)^2/4\varepsilon^2]^{1/2}} \int_{\xi_1}^{\xi_2} \frac{d\xi}{[(C_1 - \xi)(\xi + C_2)(1-\xi)\xi]^{1/2}}; \qquad (7)$$

$$\xi_{1,2} = \frac{\lambda^2 \nu (1-\nu)/4\varepsilon^2 - \chi(1-\chi)}{(1-\chi)^2 - \lambda^2 (1-\nu)^2/4\varepsilon^2} \pm \left\{ \left[\frac{\chi(1-\chi) - \lambda^2 \nu (1-\nu)/4\varepsilon^2}{\lambda^2 (1-\nu)^2/4\varepsilon^2 - (1-\chi)^2} \right]^2 - \frac{\lambda^2 \nu^2/4\varepsilon^2 - \chi^2 - \alpha^2_{mn}}{\lambda^2 (1-\nu)^2/4\varepsilon^2 - (1-\chi)^2} \right\}^{1/2}, \qquad (8)$$

$$\xi_1 = C_1, \quad \xi_2 = -C_2, \quad \xi = \sin^2\theta.$$

An elliptic integral of the first kind I,

$$I = \int_{\xi_1}^{\xi_2} \frac{d\xi}{[(C_1 - \xi)(\xi + C_2)(1-\xi)\xi]^{1/2}},$$

enters into (7). Let us find the values of ξ which are the poles of the integral I and determine the condensation points.

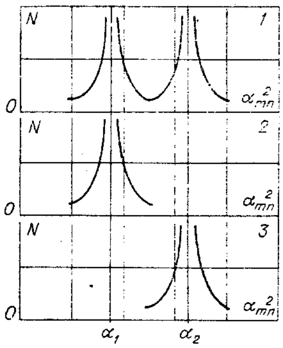

Fig. 1

Let $0 \leq \chi \leq v$ if $\chi > v$, then the x and y axes must be interchanged. Then we shall have the considered case. For such χ, v the following location of the roots C_1, C_2 is possible: a) $C_1 > 1$; $C_2 > 0$; b) $1 > C_1$; $C_2 > 0$; c) $C_1 > 1$; $0 > C_2$; d) $1 > C_1$; $0 > C_2$.

Let us examine the first case $C_1 > 1$, $C_2 > 0$. By substituting $\sin^2\varphi = (C_2 + 1)\xi/(\xi + C_2)$, we reduce the integral I to the Legendre form

$$I_1 = \int_0^{\pi/2} \frac{d\varphi}{[C_1(C_2+1) - (C_1 + C_2)\sin^2\varphi]^{1/2}}.$$

Its poles will be the points $C_1 = 0$, $C_2 = -1$,

In the second case $1 > C_1$, $C_2 > 0$ the substitution $\sin^2\varphi = (C_1 + C_2)\xi/C_1 \times (\xi + C_2)$ reduces the integral to the Legendre form

$$I_2 = \int_0^{\pi/2} \frac{d\varphi}{[(C_1 + C_2) - C_1(1 + C_2)\sin^2\varphi]^{1/2}}.$$

where the poles are the points $C_1 = 1$, $C_2 = 0$. The two remaining cases $C_1 > 1$, $0 > C_2$ and $1 > C_1$, $0 > C_2$ reduce, respectively, to the integrals I_1 and I_2 by using the substitutions

$$\sin^2\varphi = (\xi + C_2)/(1 + C_2)\xi, \quad \sin^2\varphi = C_1(\xi + C_2)/(C_1 + C_2)\xi.$$

The four poles of the integral I have been found: $C_1 = 0$ or $C_2 = -1$; $C_1 = 1$ or $C_2 = 0$. Taking (8) into account, $\xi = 0$, $\xi = 1$. Substituting these values of ξ in the exponential α^2_{mn}, we obtain two condensation points: $\alpha^2_{mn} = \alpha_1$ for $\xi = 0$ and $\alpha^2_{mn} = \alpha_2$ for $\xi = 1$. If both roots C_1, C_2 fall into any of the four domains considered above, then we have one condensation point: $\xi = 0$ or $\xi = 1$. Curves 2 and 3 in Fig. 1 correspond to this case. But it can turn out that both roots fall on the boundary of the adjacent domains, for example, $C_2 = 0$, $C_1 = 1$. Then two condensation points exist: $\xi = 0$, $\xi = 1$ (curve 1 in Fig. 1). If one of the roots falls within the domain and the other on the boundary, then two condensation points also exist (Fig. 1, curve 1) which can even coincide (Fig. 1, curves 2 and 3).

Numerical experiments [7] showed that the density function of the natural motions is a sufficiently complex function for dynamic shell loading. The singularities of this function have been investigated analytically above.

Following [1], among the motions α^2_{mn} let us extract the governing motions α^2_{mn*} for which the coefficient in the exponential achieves its greatest value. To find α^2_{mn*} we have the system

$$\partial\alpha^2_{mn}/\partial r = 0, \quad \partial\alpha^2_{mn}/\partial\xi = 0,$$

from which

$$\begin{gathered} \alpha^2_{mn*} = 2^{-1}\lambda[v + \xi(1 - v)]r^2 - [\chi + \xi(1 - \chi)]^2; \\ \lambda^2 = [\chi + \xi(1 - \chi)](1 - \chi)/[v + \xi(1 - v)](1 - v); \\ r^2 = 2^{-1}\varepsilon^{-2}\lambda[v + \xi(1 - v)]. \end{gathered} \tag{9}$$

The expressions (9) yield the governing motions. For $\xi = 0$, $\xi = 1$ the density of the governing motions tends to infinity in which case a set of motions corresponds to the exponential with maximum exponent.

Let us investigate the influence of condensation points on the nature of the motion during buckling. The location of the condensation points (Fig. 1) is determined by the shell geometry and the kind of loading. Hence, the subsequent analysis is based on an analysis of problems of buckling of a shell with a given geometry and a given kind of loading.

In the case of loading a cylindrical shell $\chi = 0$ by a constant intense load in the axial direction $v = 0$, for $\xi = 0$, $\xi = 1$ we have the two condensation points α_1 and α_2 from (5), where $\alpha_2 > 0 > \alpha_1$. Therefore, $\alpha_1 < 0$ must be discarded, since only motions with exponent $\alpha^2_{mn} > 0$ are studied. Curve 3 in Fig. 1 corresponds to this problem. Substituting $\chi = v = 0$ and $\xi = 1$ into (9), we obtain

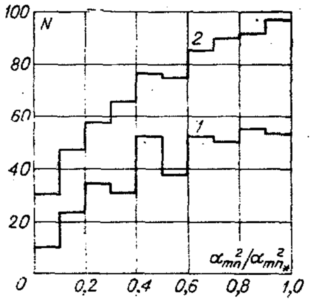

Fig. 2

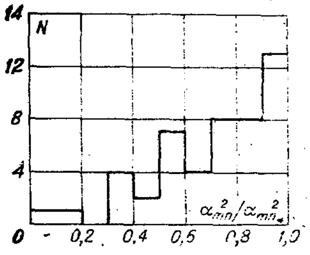

Fig. 3

$$\alpha^2_{mn*} = \alpha_2 = 4^{-1}\varepsilon^{-2} - 1.$$

Thus, the governing motions drop into the condensation point, i.e., there is an infinite set of motions with almost coincident exponents in a small neighborhood of the governing motions for $\varepsilon \to 0$.

We have an analogous result in the problem of loading a sphere by hydrostatic pressure at $\chi = v = 1$ (curve 3 in Fig. 1 corresponds to this problem): the governing motions drop into the condensation point.

Curve 1 in Fig. 1 corresponds to the case of loading a cylinder by the transverse pressure $\chi = 0$, $v > 1$. From (9) we have

$$\alpha^2_{mn*} = \alpha_2 = \lambda^2 4^{-1}\varepsilon^{-2} - 1, \quad \lambda^2 > 1.$$

The governing motions also drop into the condensation point. The existence of condensation points for the first two problems has been shown in [3]. A natural motion density function has been constructed in this work in the general case of dynamic loading of an arbitrary shell. It turns out that this function has two condensation points. Governing motions have been found and the influence of the natural motion density on them has been studied. The possibility of replacing a system with an infinite number of degrees of freedom by a system with one degree of freedom for which the exponent in the exponential is maximal is indicated in [1]. If the buckling process is studied in a finite interval, such a replacement is not always possible. In the case of the governing motions dropping into the condensation point it is necessary to take account of the natural motion density [3].

A natural motion density function having two condensation points has been constructed earlier. Results of computations confirming the existence of condensation domains in problems of cylindrical shell loading by an axial force and a transverse pressure are presented in Fig. 2. The calculations were carried out by means of (5) for given values of the parameters R/h, L/R, and the overloads λ. From the values of α^2_{mn} obtained only the positive $\alpha^2_{mn} > 0$ were selected and the maximal α^2_{mn*} was determined. The curves in Figs. 2 and 3 were constructed as follows: the interval of α^2_{mn} between 0 and α^2_{mn*} was separated into 10 equal parts, the number of motions α^2_{mn} falling into such intervals was computed, and plotted on the figure. A step function was hence obtained.

The problem of loading a cylinder by an axial force (see Fig. 2) was computed for the following values of the parameters: curve 1 is for $h/R = 1/50$, $L/R = 2$, $\alpha^2_{mn*} = 4.888$, $\lambda = 0.0605$; curve 2 is for $h/R = 1/100$, $L/R = 2$, $\alpha^2_{mn*} = 4.894$, $\lambda = 0.0302$. In the case of a cylinder under transverse pressure (Fig. 3), the following values of the parameters were selected: $h/R = 1/400$, $L/R = 2$, $\alpha^2_{mn*} = 0.188$, $\lambda = 0.0029$.

The solution (2) in the form of a sine series is not complete for a closed shell; the solution in a cosine series must still be taken into account and then the number of motions dropping into the given interval will be doubled.

It is seen from Fig. 2 that the natural motion density grows with the diminution in the thinness of the wall. A system with an infinite number of degrees of freedom (2) is approximated well by a system with a large number of degrees of freedom: in practical computations it is necessary to retain a large number of terms of the series (2), which is determined by the shell being thin-walled and by the type of loading.

The problem of cylindrical shell loading by transverse pressure theoretically yields coincidence of the governing motions with the condensation point, but the motion density is negligible (Fig. 3) for a given wall thinness on the order of $h/R \geq 1/400$ in the overloading range under investigation. In this case, the infinite series (2) can be approximated by a finite series with several terms or even with one term.

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